JOURNAL OF MULTIVARIATE ANALYSIS 9, 434–441 (1979)

On Delayed Averages of Brownian Motion in Banach Spaces

MOU-HSIUNG CHANG

The University of Alabama in Huntsville

Communicated by T. Hida

Let $\{W(t): t \geqslant 0\}$ be μ-Brownian motion in a real separable Banach space B, and let a_T be a nondecreasing function of T for which (i) $0 < a_T \leqslant T\,(T \geqslant 0)$, (ii) a_T/T is nonincreasing. We establish a Strassen limit theorem for the net $\{\xi_T: T \geqslant 3\}$, where

$$\xi_T(t) = \frac{W(T + ta_T) - W(T)}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}}, \qquad 0 \leqslant t \leqslant 1.$$

1. INTRODUCTION

Let B be a real separable Banach space with the norm $\|\cdot\|$, and let B^* be its topological dual. If μ is a mean zero Gaussian measure on B then μ is generated by a real separable Hilbert space $H_\mu \subset B$ with the norm $\|\cdot\|_\mu$. This pair of spaces (B, H_μ) is often referred to as an abstract Wiener space [3]. Perhaps one of the most important properties of the abstract Wiener space is that $\|x\| \leqslant \|\mu\|\,\|x\|_\mu$ for every x in H_μ, where $\|\mu\| = [\int_B \|x\|^2\, d\mu(x)]^{1/2}$. Consequently, through the continuous injection $i: H_\mu \to B$ and the restriction map $i^*: B^* \to H_\mu^*$, we have the relation $B^* \subset H^* \approx H_\mu \subset B$. Let $\{W(t): t \geqslant 0\}$ denote μ Brownian motion in B with the transition probability $P_t(x, A) = \mu((A - x)/t^{1/2})$. The existence of μ Brownian motion in B is known in [4]. Let C_B denote the space of continuous functions f from $[0, 1]$ into B with $f(0) = 0$. Then C_B is a real separable Banach space under the norm $\|f\|_{C_B} = \sup_{0\leqslant t\leqslant 1}\|f(t)\|$, and $\{W(t): 0 \leqslant t \leqslant 1\}$ induces a mean zero Gaussian measure P_W on the Borel measure space $(C_B, \mathscr{F})$ [5]. In the abstract Wiener space setting, P_W is generated by a real separable Hilbert space $\mathscr{H}$ which can be characterized as follows. $f \in \mathscr{H}$ if and only if $f(0) = 0$, $f(t) \in H_\mu$ for each $t \in [0, 1]$, each $x_j^*\, f \in H_0$ and $\sum_j \int_0^1 [(d/dt)\, x_j^*(f)(t)]^2\, dt < \infty$, where $\{x_j^*: j \leqslant 1\} \subset B^*$ is such that the set

Received July 26, 1978.

AMS 1970 subject classifications: Primary 60J65, 60G15, 60G17; Secondary 28A40.

Key words and phrases: Gaussian measures, abstract Wiener space, Brownian motion in Banach spaces, Strassen law of the iterated logarithm.

0047-259X/79/030434-08$02.00/0

$\{x_j: j \geqslant 1\}$, $x_j = \int_B x_j^*(x)x\, d\mu(x)$ forms a complete orthogonormal set of H_μ, and H_0 is the space of real absolutely continuous functions ϕ on [0, 1] with $\phi(0) = 0$ and $\int_0^1 [\phi'(t)]^2\, dt < \infty$. The inner product and the closed unit ball of $\mathscr{H}$ are denoted by $(f_1, f_2)_{\mathscr{H}} = \sum_j \int_0^1 [(d/dt)\, x_j^*(f_1)(t)(d/dt)\, x_j^*(f_2)(t)]\, dt$ and $K = \{f \in \mathscr{H}: \|f\|_{\mathscr{H}} \leqslant 1\}$, respectively. It is known that K is compact in C_B [5].

The main result of this paper is the following type limit theorem.

THEOREM 1. *Let $\{W(t): t \geqslant 0\}$ be μ Brownian motion in B. Suppose a_T is a nondecreasing function of T for which*

(i) $0 < a_T \leqslant T$ $(T \geqslant 0)$,

(ii) a_T/T *is nonincreasing.*

Let

$$\xi_T(t) = \frac{W(T + ta_T) - W(T)}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}}, \qquad 0 \leqslant t \leqslant 1.$$

Then, with probability one, the net $\{\xi_T: T \geqslant 3\}$ is relatively compact in C_B and the set of its limit points coincides with K.

When $a_T = T$, Theorem 1 obviously reduces to Strassen's law of itereated logarithm obtained by LePage and Kuelbs [5]. When $B = R$, Theorem 1 also includes Lai's results [6]. These special cases will be discussed under Corollaries 1 and 2.

This work is inspired by a paper of Csöorgő and Révész [1] in which the result of increments of the following type,

$$\varlimsup_{T\to\infty} \sup_{0 \leqslant t \leqslant T - a_T} \frac{|W(t + a_T) - W(t)|}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}} = 1 \quad \text{w.p.1.}$$

is obtained for standard Brownian motion $\{W(t): t \geqslant 0\}$.

2. SOME LEMMAS

For $\epsilon > 0$, let K_ϵ denote the uniform ϵ-neighborhood of K.

LEMMA 1. *For each $\epsilon > 0$, there exists $r > 1$ such that*

$$P_W\{f \in C_B: f/(2[\log(T/a_T) + \log\log T])^{1/2} \notin K_\epsilon\} \leqslant \exp\{-r^2 \log\log T\}$$

for all sufficiently large T.

Proof. For $k \geqslant 1$, let $f^{(k)}(t) = \sum_{j=1}^{k} x_j^*(f(t))\, x_j$ for $t \in [0, 1]$. Then to prove the lemma, we simply note that

$$P_W \left\{ f \in C_B: \frac{1}{r_0} f^{(k)} \Big/ \left(2\left[\log\left(\frac{T}{a_T}\right) + \log\log T\right]\right)^{1/2} \notin K \right\}$$
$$\leqslant P_W \left\{ f \in C_B: \frac{1}{r_0} f^{(k)}/(2\log\log T)^{1/2} \notin K \right\}$$

and

$$P_W \left\{ f \in C_B: \| f - f^{(k)} \|_{C_B} \geqslant \frac{\epsilon}{2}\left(2\left[\log\left(\frac{T}{a_T}\right) + \log\log T\right]\right)^{1/2} \right\}$$
$$\leqslant P_W \left\{ f \in C_B: \| f - f^{(k)} \|_{C_B} \geqslant \frac{\epsilon}{2}(2\log\log T)^{1/2} \right\}$$

for each $k \geqslant 1$ and $r_0 > 1$. The rest of the proof follows exactly as that of Proposition 1 in [7, p. 106].

LEMMA 2. *If $\epsilon > 0$ one may choose $c > 1$ sufficiently close to one so that for every $f \in C_B$ the statements $[c^n] \leqslant T \leqslant [c^{n+1}]$ and $f(a_{[c^{n+1}]}\, \cdot)/\beta_{[c^{n+1}]} \in K_\epsilon$ together imply $f(a_T\, \cdot)/\beta_T \in K_{2\epsilon}$ for all sufficiently large n, where*

$$\beta_T = \{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}.$$

Proof. Note that $1 \leqslant \beta_{[c^{n+1}]}/\beta_{[c^n]} \leqslant c$ if n is sufficiently large. Then the lemma follows exactly as that of Lemma 6 in [5, p. 260].

For any positive integer m, define $\{\tilde{W}_m(t): 0 \leqslant t \leqslant 1\}$ as

$$\tilde{W}_m(i/m) = W(i/m) \qquad \text{for} \quad i = 1, 2, \ldots, m;$$

and

$$\tilde{W}_m(t) \quad \text{is linear on the interval} \quad [(i-1)/m, i/m] \qquad \text{for} \quad i = 1, 2, \ldots, m.$$

Let

$$\tilde{\xi}_{m,T}(t) = \frac{\tilde{W}_m(T + ta_T) - \tilde{W}_m(T)}{\{2a_T[\log(T/a_T) + \log\log T]^{1/2}\}}, \qquad 0 \leqslant t \leqslant 1.$$

From [2], it is known that there exits $\alpha > 0$ such that $E[\exp \alpha \| W(1) \|^2] < \infty$. We have the following lemma.

LEMMA 3. *Let $\epsilon > 0$, $c > 1$, and $\alpha > 0$ be such that $E[\exp \alpha \| W(1) \|^2] < \infty$. If m is a positive integer with $2\alpha m \epsilon^2 > 1$, then*

$$\Pr\{\| \tilde{\xi}_{m,[c^n]} - \xi_{[c^n]} \|_{C_B} \leqslant \epsilon \text{ for all but finitely many } n\} = 1.$$

Proof. Temporarily, fix j and m and ϵ. For $i = 1, 2,\ldots, m$, let A_i be the event

$$A_i = \{\sup_{(i-1)/m\leqslant t\leqslant i/m} \| \tilde{\xi}_{m,j}(t) - \xi_j(t)\| > \epsilon\}.$$

Then

$$\{\| \tilde{\xi}_{m,j} - \xi_j \|_{C_B} > \epsilon\} = \bigcup_{i=1}^{m} A_i .$$

By stationary independence increments of $\{W(t): t \geqslant 0\}$, we have $\Pr(A_1) = \cdots = \Pr(A_m)$. Now it is clear that

$$A_1 = \{\sup_{0\leqslant s\leqslant a_j/m} \| W(s) - (m/a_j)\, sW(a_j/m)\| > \epsilon\beta_j\},$$

where β_T is as that of Lemma 2. Note that $A_1 \subset C \cup D$, where

$$C = \{\sup_{0\leqslant s\leqslant a_j/m} \| W(s)\| > \epsilon\beta_j\}$$

and

$$D = \{\sup_{0\leqslant s\leqslant a_j/m} \| W(s) - W(a_j/m)\| > \epsilon\beta_j\}.$$

This is because $0 \leqslant s \leqslant a_j/m$ and $W(s) - (m/a_j)s\, W(a_j/m)$ is a convex combination of $W(s)$ and $W(a_j/m)$. It is also known that $\Pr(C) = \Pr(D)$. Now if $\alpha > 0$ is such that $E[\exp \alpha \| W(1)\|^2] < \infty$, we have, by Chevyshev's inequality, that

$$\begin{aligned}
\Pr(C) &\leqslant 2\Pr\{\| W(1)\| > \epsilon(2m[\log(j/a_j) + \log\log j])^{1/2}\} \\
&\leqslant 2\Pr\{\| W(1)\| > \epsilon(2m \log\log j)^{1/2}\}, \\
&\leqslant r \exp\{-2\alpha m\epsilon^2 \log\log j\},
\end{aligned}$$

where r is a constant independent of j. Set $j = [c^n]$. Then, since $2\alpha m\epsilon^2 = \theta > 1$,

$$\sum_n \Pr\{\| \tilde{\xi}_{m,[c^n]} - \xi_{[c^n]} \|_{C_B} > \epsilon\} \sim \sum_n \frac{1}{n^\theta} < \infty.$$

The lemma follows by Borel–Cantelli's lemma.

3. Proof of Theorem 1 and Corollaries

The proof of Theorem 1 consists of proving the following two parts:

(a) $\Pr\{\lim_{T\to\infty} \| \xi_T - K \|_{C_B} = 0\} = 1.$

(b) Given $\epsilon > 0$ and $h \in K$ with $\| h \|_{\mathscr{H}} < 1$, there is a $c > 1$ so that

$$\Pr\{\| \xi_{[c^n]} - h \|_{C_B} < \epsilon \text{ for infinitely many } n\} = 1.$$

The proof of (a) follows exactly as that of Theorem 1 in [5, p. 261] by using Lemmas 1 and 2.

To prove (b), we choose (and then fix) a sufficiently large k such that $\| h - h^{(k)} \|_{C_B} < \epsilon/3$ and

$$\Pr\{\| \xi_T - \xi_T^{(k)} \|_{C_B} < \epsilon/3 \text{ for all sufficiently large } T\} = 1.$$

The above are permissible by Lemma 4(d) in [5, p. 257].

Let m be a positive integer such that $\alpha m\epsilon/18 \| \mu \|^2 > 1$. For $i = 1, 2,..., m$ and $j = 1, 2,..., k$, let $a_{ij} = h_j^{(k)}(i/m) - h_j^{(k)}[(i-1)/m]$, where $h_j^{(k)}$ means the jth-component function of $h^{(k)}$.

Since $h \in K$ with $\| h \|_{\mathscr{H}} < 1$, we have $m \sum_{i=1}^m \sum_{j=1}^k a_{ij}^2 < 1$. Now choose a positive δ such that $\delta < \epsilon/6m \| \mu \|$ and $\theta = m \sum_{i=1}^m \sum_{j=1}^k (a_{ij} \pm \delta/k)^2 < 1$. Here

$$\begin{aligned} a_{ij} \pm \delta/k &= a_{ij} + \delta k \qquad \text{when} \quad a_{ij} \geqslant 0 \\ &= a_{ij} - \delta/k \qquad \text{when} \quad a_{ij} < 0. \end{aligned}$$

We define the event

$$A_n = \left\{ \left| \xi_{j,[c^n]}^{(k)} \left(\frac{i}{m} \right) - \xi_{j,[c^n]}^{(k)} \left(\frac{i-1}{m} \right) - a_{ij} \right| < \delta/k \right.$$

for $i = 1, 2,..., m$ and $j = 1, 2,..., k\}$. Then,

$$\Pr(A_n) \geqslant \prod_{i=1}^m \prod_{j=1}^k \frac{1}{(2\pi)^{1/2}} \int_{m^{1/2}(a_{ij}-\delta/k)\alpha_{[c_n]}}^{m^{1/2}(a_{ij}+\delta/k)\alpha_{[c_n]}} \exp(-x^2/2)\, dx,$$

where $\alpha_T = (2[\log(T/a_T) + \log\log T])^{1/2}$. Using the estimate that

$$\frac{1}{(2\pi)^{1/2}} \int_a^b \exp\{-x^2/2\}\, dx \geqslant \frac{(b-a)}{(2\pi)^{1/2}} \exp\{-\tfrac{1}{2} \max(a^2, b^2)\}$$
$$\text{for } -\infty < a < b < \infty,$$

we have, for some constant $\gamma > 0$ (whose value may vary from line to line), that

$$\Pr(A_n) \geqslant \gamma\alpha_{[c^n]}^{mk} \prod_{i=1}^{m} \prod_{j=1}^{k} \exp\left\{-\frac{m}{2}(a_{ij} \pm \delta/k)^2 \alpha_{[c^n]}^2\right\}$$

$$= \gamma\alpha_{[c^n]}^{mk} \exp\left\{-\frac{m}{2}\sum_{i=1}^{m}\sum_{j=1}^{k}(a_{ij} \pm \delta/k)^2 \alpha_{[c^n]}^2\right\}$$

$$= \gamma\alpha_{[c^n]}^{mk} \exp\left\{-\theta \log\left(\frac{[c^n]}{a_{[c^n]}} \cdot \log[c^n]\right)\right\}$$

$$= \gamma\left[2\log\left(\frac{[c^n]}{a_{[c^n]}} \cdot \log[c^n]\right)\right)^{mk/2} \Big/ \left(\frac{[c^n]}{a_{[c^n]}}\right)^{\theta} (\log[c^n])^{\theta}$$

$$\geqslant \frac{\gamma a_{[c^n]} \log\log[c^n]}{[c^n]\log[c^n]} \geqslant \frac{\gamma \log\log[c^n]}{[c^n]\log[c^n]}.$$

Hence for $c = m$, we have $A_1, A_2, \ldots,$ independent. Now the function $(\log\log x)/(x\log x)$ is continuous, decreasing, and positive. By the integral test, we have $\sum_{n=1}^{\infty}\Pr(A_n) = \infty$. Therefore, $\Pr(\limsup_n A_n) = 1$ by Borel–Cantelli's lemma.

Now,

$$\sup_{0\leqslant t\leqslant 1} \| \xi_{[c^n]}^{(k)}(t) - h^{(k)}(t)\|_\mu$$
$$\leqslant \sup_{0\leqslant t\leqslant 1} \| \xi_{[c^n]}^{(k)}(t) - \bar{\xi}_{m,[c^n]}^{(k)}(t)\|_\mu + \sup_{0\leqslant t\leqslant 1} \| \bar{\xi}_{m,[c^n]}^{(k)}(t) - h^{(k)}(t)\|_\mu .$$

By Lemma 3, we have, for $m > 18\|\mu\|^2/\alpha\epsilon^2$,

$$\Pr\{\sup_{0\leqslant t\leqslant 1} \| \xi_{[c^n]}^{(k)}(t) - \bar{\xi}_{m,[c^n]}^{(k)}(t)\|_\mu < \epsilon/6\|\mu\| \text{ for all sufficiently large } n\} = 1.$$

As for the second term in the last inequality, we have

$$\sup_{0\leqslant t\leqslant 1} \| \bar{\xi}_{m,[c^n]}^{(k)}(t) - h^{(k)}(t)\|_\mu = \max_i \left\| \bar{\xi}_{m,[c^n]}^{(k)}\left(\frac{i}{m}\right) - h^{(k)}\left(\frac{i}{m}\right)\right\|_\mu$$

$$= \max_i \left\| \xi_{[c^n]}^{(k)}\left(\frac{i}{m}\right) - h^{(k)}\left(\frac{i}{m}\right)\right\|_\mu$$

$$\leqslant \max_i \sum_{j=1}^{k} \left| \xi_{j,[c^n]}^{(k)}\left(\frac{i}{m}\right) - h_j^{(k)}\left(\frac{i}{m}\right)\right|$$

$$< \sum_{i=1}^{m}\sum_{j=1}^{k} \left| \xi_{j,[c^n]}^{(k)}\left(\frac{i}{m}\right) - \xi_{j,[c^n]}^{(k)}\left(\frac{i-1}{m}\right) - a_{ij}\right|,$$

where we use the convexity in line 1, the definition of $\bar{\xi}$ in line 2. Therefore $A_n \subset B_n$, where $B_n = \{\sup_{0\leqslant t\leqslant 1} \| \bar{\xi}_{m,[c^n]}^{(k)}(t) - h^{(k)}(t)\|_\mu < \epsilon/6\|\mu\|\}$. Consequently, $\Pr\{\limsup_n B_n\} = 1$. This proves that $\Pr\{\|\xi_{[c^n]} - h\|_{C_B} < \epsilon$ for infinitely many $n\} = 1$, since

$$\|\xi_{[c^n]} - h\|_{C_B} \leqslant \|\xi_{[c^n]} - \xi^{(k)}_{[c^n]}\|_{C_B} + \|h - h^{(k)}\|_{C_B} + \|\xi^{(k)}_{[c^n]} - h^{(k)}\|_{C_B}$$

$$< \frac{2\epsilon}{3} + \|\mu\| \sup_{0\leqslant t\leqslant 1} \|\xi^{(k)}_{[c^n]}(t) - h^{(k)}(t)\|_\mu$$

$$< \frac{2\epsilon}{3} + \|\mu\|\{\epsilon/6\|\mu\| + \epsilon/6\|\mu\|\} = \epsilon.$$

Corollaries 1 and 2 are results obtained by LePage and Kuelbs [5] and Lai [6], respectively.

COROLLARY 1. *The net*

$$\left\{\frac{W(T\cdot)}{(2T\log\log T)^{1/2}} : T \geqslant 3\right\}$$

is relatively compact in C_B and the set of its limit points coincides with K.

Proof. Let $a_T = T$.

COROLLARY 2. (i) *If $a_T = o(T^\beta)$ for any $\beta > 0$, then with probability one the net*

$$\left\{\frac{W(T + \cdot a_T) - W(T)}{(2a_T \log T)^{1/2}} : T \geqslant 3\right\}$$

is relatively compact in C_B and the set of its limit points coincides with K.

(ii) *Let $0 < \alpha < 1$. If $a_T = T^\alpha\psi(T)$ with $\psi(T) + (\psi(T))^{-1} = o(T^\beta)$ for any $\beta > 0$ and $\lim_{p\to 1}\lim_{T\to\infty} a_{pT}/a_T = 1$, then with probability one the net*

$$\left\{\frac{W(T + \cdot a_T) - W(T)}{[2(1-\alpha)\, a_T \log T]^{1/2}} : T \geqslant 3\right\}$$

is relatively compact in C_B and the set of its limit points coincides with K.

(iii) *Let $\beta > 0$. If $a_T = \beta T(1 + o(1))$, then with probability one the net*

$$\left\{\frac{W(T + \cdot a_T) - W(T)}{(2a_T \log\log T)^{1/2}} : T \geqslant 3\right\}$$

is relatively compact in C_B and its set of its limit points coincides with K.

(iv) *Let* $\lambda > 0, \beta > 0$. *If* $a_T = \lambda T(\log T)^{-\beta}(1 + o(1))$, *then with probability one the net*

$$\left\{\frac{W(T + \cdot a_T) - W(T)}{[2(1 + \beta)\, a_T \log\log T]^{1/2}} : T \geqslant 3\right\}$$

is relatively compact in C_B *and the set of its limit points coincides with* K.

Proof. (i) If $a_T = o(T^\beta)$ for any $\beta > 0$. Then $\log(T/a_T) + \log\log T \sim \log T$ for all large T.

(ii) If $a_T = T^\alpha \psi(T)$ with $\psi(T) + (\psi(T))^{-1} = o(T^\beta)$ for any $\beta > 0$ and $\lim_{p\to 1} \lim_{T\to\infty} a_{pT}/a_T = 1$, then $\log(T/a_T) + \log\log T \sim \log T^{(1-\alpha)} = (1 - \alpha) \log T$ for all large T.

(iii) If $a_T = \beta T(1 + o(1))$ then $\log(T/a_T) + \log\log T \sim \log\log T$ for all large T.

(iv) If $a_T = \lambda T(\log T)^{-\beta}(1 + o(1))$ then $\log(T/a_T) + \log\log T \sim (1 + \beta) \log\log T$ for all large T.

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